J. Fluid Mech. (1987), *vol.* 181, *pp.* 381–386
Printed in Great Britain

Note on a Hamiltonian formalism for the flow of a magnetic fluid with a free surface

By O. SERO-GUILLAUME AND D. BERNARDIN
Laboratoire D'Energetique et de Mecanique Theorique et Appliquee, 24–30, rue Lionnois, 54013 Nancy Cedex, France

(Received 17 February 1986 and in revised form 21 January 1987)

The potential flow of an inviscid ferromagnetic fluid is formulated as a Hamiltonian system: the conjugate coordinates are the normal displacement of the free surface and the velocity potential on the free surface. If we take the equation of the magnetic field as a constraint, we can see that the flow of a magnetic liquid is a control-theory problem; the potential of the magnetic self-field is the control variable.

1. Introduction

A magnetic fluid is a homogeneous suspension of ferrite particles in a solvent and it behaves like a normal fluid except that it can experience magnetic force. It is a good insulator, so the interactions of magnetic fields with currents of free charge can be taken as negligible. When the magnetic fluid is placed in an external magnetic field, a self-field and magnetic force appear in it; if the surface of the fluid is free, the shape of the volume is deformed; therefore the self-field and the magnetic force are modified. In previous work we derived a variational principle for the study of the equilibrium of a magnetic fluid which is applicable to a large class of problems, such as an electrized meniscus Joffre (1984) or the levitation melting process Sneyd & Moffat (1982), but such a method seemed to be unusable for a non-static process. We propose here, via Hamiltonian formalism, a generalization of the method for dynamic processes. Some studies have been devoted to variational principles for surface waves, for example Luke (1967), Miles (1977), Milder (1977), but the variations of the Lagrangian or Hamiltonian were taken along a vertical axis z; that is, if $z = f(x, y, t)$ is the equation of the interface, the variations are δf. As this method is not possible for any configuration, see Benjamin & Olver (1982), to obtain information about the general case, we propose to take variations along the normal of the surface.

2. Dynamic and magnetic equations

We shall consider a simple model for a magnetic fluid, see Cowley & Rosensweig (1967); for further discussion of models the reader can consult Brancher (1980) or Rosensweig (1985). The magnetic fluid is in the volume Ω_1, Ω_2 is the exterior of Ω_1, and it is submitted to an external magnetic field $\boldsymbol{H}_0$; the total magnetic field is $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{h}$, where $\boldsymbol{h}$ is the self-field produced by the magnetic fluid; we shall neglect hysteresis, hydrodynamic relaxation of ferrite particles and consider that the magnetization $\boldsymbol{M}$ is parallel to $\boldsymbol{H}$, i.e. $\boldsymbol{M} = \chi(H)\,\boldsymbol{H}$.

If z is the vertical coordinate and if the fluid is inviscid, Newton's law gives

$$\rho \frac{\mathrm{D}\boldsymbol{V}}{\mathrm{D}t} = -\nabla(p+\rho g z) + \mu_0 \boldsymbol{M}\cdot\nabla \boldsymbol{H}. \tag{2.1 a}$$

Conservation of volume is written

$$\nabla \cdot V = 0, \tag{2.1b}$$

and the surface strain density is

$$T = -p\hat{n} - \tfrac{1}{2}\mu_0 (M \cdot n)^2 \hat{n}, \tag{2.2}$$

where $\hat{n}$ is the unit normal vector. In (2.2) $M \cdot n$ is the scalar product, and we note that $M \cdot \hat{n} = M_n$ and, as M and H are parallel,

$$M \cdot \nabla H = \nabla \left(\int_0^H M(y)\, \mathrm{d}y \right).$$

As the initial flow is supposed to be potential, (2.1 b) reduces to

$$\nabla^2 \phi = 0. \tag{2.3}$$

Equation (2.1 a) gives the Bernoulli relation

$$\rho \frac{\partial \phi}{\partial t} + \tfrac{1}{2}\rho |\nabla \phi|^2 + p + \rho g z - \mu_0 \int_0^H M(y)\, \mathrm{d}y = k(t), \tag{2.4}$$

Using Laplace's law and (2.2) on the boundary S of Ω_1 we obtain

$$\rho \frac{\partial \phi}{\partial t} + \tfrac{1}{2}\rho |\nabla \phi|^2 + \rho g z - \tfrac{1}{2}\mu_0 M_n^2 - \mu_0 \int_0^H M(y)\, \mathrm{d}y + \sigma C = k(t), \tag{2.5}$$

where C is the mean curvature of S and $k(t)$ is a function of time. We must consider the kinematic condition on S: the normal velocity of the interface is equal to the normal velocity of the fluid. If the normal displacement is $\theta \hat{n}$, with θ a real function defined on S, the velocity of the interface is $(\partial/\partial t)(\theta \hat{n})$ and the normal velocity is $(\partial/\partial t)(\theta \hat{n}) \cdot \hat{n}$.

But

$$\frac{\partial}{\partial t}(\theta \hat{n})\, \hat{n} \cdot = \left(\frac{\partial \theta}{\partial t} \hat{n} + \theta \frac{\partial \hat{n}}{\partial t} \right) \cdot \hat{n} = \frac{\partial \theta}{\partial t},$$

because $\hat{n} \cdot \partial \hat{n}/\partial t = 0$. Thus the kinematic relation is

$$\frac{\partial \theta}{\partial t} = \nabla \phi \cdot \hat{n} = \frac{\partial \phi}{\partial \boldsymbol{n}}. \tag{2.6}$$

Now let $H_i = H_0 + h_i$, where h_i is the field due to the presence of the magnetic fluid; then $\nabla \wedge h_i = 0$, and therefore $h_i = \nabla u_i$ in Ω_i. The magnetic induction is $B_1 = \mu_0(H_1 + M(H_1))$ in Ω_1 and $B_2 = \mu_0 H_2$ in Ω_2 and B_i satisfy $\nabla \cdot B_i = 0$; the normal component of B and the tangential component of H are conserved, i.e. $H_{2n} = H_{1n} + M_n$ and $H_{1t} = H_{2t}$ on S; the second relation is equivalent to $u_1 = u_2$ on S.

Let E_m be the magnetic energy:

$$E_\mathrm{m} = \mu_0 \int_{\Omega_1} \left(H_1^2 + 2\int_0^{H_1} M(y)\, \mathrm{d}y \right) \mathrm{d}\Omega + \tfrac{1}{2}\mu_0 \int_{\Omega_2} H_2^2\, \mathrm{d}\Omega. \tag{2.7}$$

A direct calculation of the first variation of E_m gives

$$\delta E_\mathrm{m} = \mu_0 \int_{\Omega_1} (H_1 + M(H_1)) \cdot \delta H_1\, \mathrm{d}\Omega + \int_{\Omega_2} H_2 \cdot \delta H_2\, \mathrm{d}\Omega. \tag{2.8}$$

We note that $\delta \boldsymbol{H}_i = \boldsymbol{\nabla}\delta u_i$; then

$$\delta E_{\mathrm{m}} = \mu_0 \int_S (\boldsymbol{H}_1 + \boldsymbol{M}(H_1))\cdot\hat{\boldsymbol{n}}\,\delta u_1\,\mathrm{d}S - \mu_0 \int_S \boldsymbol{H}_2\cdot\hat{\boldsymbol{n}}\,\delta u_2\,\mathrm{d}S$$

$$-\mu_0 \int_{\Omega_1} \boldsymbol{\nabla}\cdot(\boldsymbol{H}_1 + \boldsymbol{M}(H_1))\,\delta u_1\,\mathrm{d}\Omega - \mu_0 \int_{\Omega_2} \boldsymbol{\nabla}\cdot\boldsymbol{H}_2\,\delta u_2\,\mathrm{d}\Omega. \quad (2.9)$$

Therefore the relation

$$\delta E_{\mathrm{m}} = 0 \quad \text{for any } \delta u \quad (2.10)$$

gives the magnetic equations; the relation (2.10) implies that (2.7) is a correct form of the magnetic energy.

3. Hamiltonian formulation and discussion

The Hamiltonian $\mathscr{H}$ involves the other energies

$$U_{\mathrm{g}} = \int_{\Omega_1} \rho g z\,\mathrm{d}\Omega, \quad \mathrm{u}_\sigma = \sigma \int_S \mathrm{d}S, \quad K = \tfrac{1}{2}\rho \int_{\Omega_1} |\boldsymbol{\nabla}\phi|^2\,\mathrm{d}\Omega, \quad (3.1)$$

which are gravitational energy, interfacial energy and kinetic energy respectively. Thus the Hamiltonian is

$$\mathscr{H} = \frac{1}{\rho}(K + U_{\mathrm{g}} + U_\sigma - E_{\mathrm{m}}). \quad (3.2)$$

The minus sign in (3.2) is because the currents in the external coils remain at constant amplitude, see Sneyd & Moffatt (1982).

Suppose that Ω_1 is perturbed by a small displacement $\delta\theta\hat{\boldsymbol{n}}$ normal to S; then the changes in U_{g} and U_σ are

$$\delta U_{\mathrm{g}} = \rho g \int_S z\,\delta\theta\,\mathrm{d}S, \quad \delta U_\sigma = \sigma \int_S C\,\delta\theta\,\mathrm{d}S, \quad (3.3)$$

and the change in E_{m} is

$$\delta E_{\mathrm{m}} = \mu_0 \int_{\delta\Omega_1} \left(\tfrac{1}{2}H_1^2 + \int_0^{H_1} M(y)\,\mathrm{d}y\right)\mathrm{d}\Omega + \mu_0 \int_{\delta\Omega_2} \tfrac{1}{2}H_1^2\,\mathrm{d}\Omega$$

$$+\mu_0 \int_{\Omega_1} (\boldsymbol{H}_1 + \boldsymbol{M}(H_1))\cdot\delta\boldsymbol{H}_1\,\mathrm{d}\Omega + \mu_0 \int_{\Omega_2} \boldsymbol{H}_2\cdot\delta\boldsymbol{H}_2\,\mathrm{d}\Omega. \quad (3.4)$$

Now, for simplicity, let $\boldsymbol{H} = \boldsymbol{\nabla}\Phi$, then

$$\delta E_{\mathrm{m}} = \mu_0 \int_S \left(\tfrac{1}{2}H_1^2 + \int_0^{H_1} M(y)\,\mathrm{d}y - \tfrac{1}{2}H_2^2\right)\delta\theta\,\mathrm{d}S + \mu_0 \int_S (H_{1n} + M_n)\,\delta\Phi_1\,\mathrm{d}S - \mu_0 \int_S H_{2n}\,\delta\Phi_2\,\mathrm{d}S.$$

But as the tangential component of $\boldsymbol{H}$ is conserved through S, $\Phi_1 - \Phi_2 = 0$ on $S + \delta S$, and the first variation is

$$\delta\Phi_1 - \delta\Phi_2 + \delta\theta\left(\frac{\partial\Phi_1}{\partial n} - \frac{\partial\Phi_2}{\partial n}\right) = 0.$$

We recall that

$$\frac{\partial\Phi_1}{\partial n} - \frac{\partial\Phi_2}{\partial n} = H_{1n} - H_{2n} = -M_n,$$

then we obtain $\delta\Phi_1 - \delta\Phi_2 = \delta\theta M_n$ on S, and

$$\delta E_{\mathrm{m}} = \mu_0 \int_S \left(\tfrac{1}{2}H_1^2 - \tfrac{1}{2}H_2^2 + (H_{1n} + M_n)\,M_n + \int_0^{H_1} M(y)\,\mathrm{d}y\right)\delta\theta\,\mathrm{d}s,$$

but $H_1^2 = H_{1n}^2 + H_{1t}^2$, $H_2^2 = H_{2n}^2 + H_{2t}^2$ and $H_{1t} = H_{2t}$; thus

$$\delta E_{\mathrm{m}} = \mu_0 \int_S \left(\tfrac{1}{2}M_n^2 + \int_0^{H_1} M(y)\,\mathrm{d}y\right)\delta\theta\,\mathrm{d}S. \tag{3.5}$$

The variation of kinetic energy is given by (see the Appendix)

$$\delta K = \tfrac{1}{2}\rho \int_S |\nabla\phi|^2\,\delta\theta\,\mathrm{d}S - \rho\int_S \delta\theta\left(\frac{\partial\phi}{\partial n}\right)^2 \mathrm{d}S. \tag{3.6}$$

We now multiply the relation (2.5) by $\delta\theta$ and integrate over S:

$$\int_S \frac{\partial\phi}{\partial t}\delta\theta\,\mathrm{d}S = -\int_S\left(\tfrac{1}{2}|\nabla\phi|^2 + gz - \frac{\mu_0}{2\rho}M_n^2 - \frac{\mu_0}{\rho}\int_0^{H_1} M(y)\,\mathrm{d}y + \frac{\sigma}{\rho}C\right)\delta\theta\,\mathrm{d}S. \tag{3.7}$$

We note that $\int_S \delta\theta\,\mathrm{d}S = 0$ because the volume is conserved.

We can recognize in the right-hand side of (3.7) a part of $\delta\mathscr{H}$, that is

$$\int_S \frac{\partial\phi}{\partial t}\delta\theta\,\mathrm{d}S = -\delta\mathscr{H} - \int_S\left(\frac{\partial\phi}{\partial n}\right)^2\delta\theta\,\mathrm{d}S. \tag{3.8}$$

Using the kinematic relation (2.6) the left-hand side of (3.8) is

$$\int_S\left(\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial n}\frac{\partial\theta}{\partial t}\right)\delta\theta\,\mathrm{d}S.$$

But if ϕ_S is the value of ϕ on S, the time derivative of ϕ_S is

$$\frac{\partial\phi_S}{\partial t} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial n}\frac{\partial\theta}{\partial t};$$

thus

$$\int_S\left(\frac{\partial\phi_S}{\partial t}\right)\delta\theta\,\mathrm{d}S = -\delta\mathscr{H}. \tag{3.9}$$

Therefore we obtain Hamilton's first equation

$$\frac{\partial\phi_S}{\partial t} = -\frac{\partial\mathscr{H}}{\partial\theta}. \tag{3.10}$$

Now let $\delta\phi_S$ be a variation of ϕ_S; then the variation of $\mathscr{H}$ is

$$\delta\mathscr{H} = \delta K = \int_{\Omega_1}\nabla\delta\phi\cdot\mathrm{d}\Omega = \int_S \delta\phi_S\cdot\frac{\partial\phi}{\partial n}\mathrm{d}S. \tag{3.11}$$

But since $\partial\phi/\partial n = \partial\theta/\partial t$ on S,

$$\delta\mathscr{H} = \int_S \delta\phi_S\frac{\partial\theta}{\partial t}\mathrm{d}S. \tag{3.12}$$

The relation (3.12) can be written

$$\frac{\partial\theta}{\partial t} = \frac{\partial\mathscr{H}}{\partial\phi_S}, \tag{3.13}$$

which is Hamilton's second equation.

The relations (3.10) and (3.13) show that the flow is Hamiltonian and that the conjugate variables are θ and ϕ_S; ϕ_S is the momentum and θ the position. The result is still applicable to the problems cited in §1. We note that Mestel (1983), in calculating the shape of a levitated liquid metal, introduced the potential flow induced by deformation of the volume. This method might seem to be somewhat different from the variational method of Sneyd & Moffatt, but the Hamiltonian structure of the problem shows that the two methods are equivalent.

Appendix. Variation of the kinetic energy

If $\delta\boldsymbol{\theta}$ is the vector displacement of a point $\boldsymbol{x}$ during the deformation of the volume Ω_1, the new position of the point is $\boldsymbol{x}+\delta\boldsymbol{\theta}$, and the Jacobian matrix of this transformation is

$$\boldsymbol{J} = \nabla(\boldsymbol{x}+\delta\boldsymbol{\theta}) = \boldsymbol{I}+\nabla(\delta\boldsymbol{\theta}),$$

where $\boldsymbol{I}$ is the identity matrix. To obtain the variation of K we can proceed by a change of variables; that is, if $\Omega_1+\delta\Omega_1$ is the deformed volume then

$$K+\delta K = \tfrac{1}{2}\rho\int_{\Omega_1+\delta\Omega_1}|\nabla\phi|^2\,\mathrm{d}\Omega = \tfrac{1}{2}\rho\int_{\Omega_1}|{}^{\mathrm{t}}\boldsymbol{J}^{-1}\nabla\phi|^2\det\boldsymbol{J}\,\mathrm{d}\Omega. \qquad \text{(A 1)}$$

We now expand to first order in $\delta\boldsymbol{\theta}$ the integral in the right-hand side of (A 1). We have

$$(\det\boldsymbol{J}) = 1+\nabla\cdot(\delta\boldsymbol{\theta})+o(\delta\boldsymbol{\theta})$$

and

$${}^{\mathrm{t}}\boldsymbol{J}^{-1} = \boldsymbol{I}-{}^{\mathrm{t}}\nabla(\delta\boldsymbol{\theta})+o(\delta\boldsymbol{\theta});$$

so

$$\delta K = \tfrac{1}{2}\rho\int_{\Omega_1}|\nabla\phi|^2\nabla\cdot(\delta\boldsymbol{\theta})\,\mathrm{d}\Omega-\rho\int_{\Omega_1}({}^{\mathrm{t}}\nabla(\delta\boldsymbol{\theta})\cdot\nabla\phi)\cdot\nabla\phi\,\mathrm{d}\Omega. \qquad \text{(A 2)}$$

We now verify that

$$|\nabla\phi|^2\nabla\cdot(\delta\boldsymbol{\theta})-2\nabla\phi\cdot({}^{\mathrm{t}}\nabla(\delta\theta)\cdot\nabla\phi) = \nabla\cdot\{|\nabla\phi|^2\,\delta\boldsymbol{\theta}\}-2\nabla\phi\cdot\nabla(\nabla\phi\cdot\delta\boldsymbol{\theta})).$$

Therefore $\delta\boldsymbol{K}$ can be written

$$\delta K = \tfrac{1}{2}\rho\int_{\Omega_1}\nabla\cdot(|\nabla\phi|^2\,\delta\boldsymbol{\theta})\,\mathrm{d}\Omega-\rho\int_{\Omega_1}\nabla\phi\cdot(\nabla(\nabla\phi\cdot\delta\boldsymbol{\theta}))\,\mathrm{d}\Omega,$$

which gives the surface integral, as $\nabla^2\phi = 0$,

$$\delta K = \tfrac{1}{2}\rho\int_S|\nabla\phi|^2\,\delta\boldsymbol{\theta}\cdot\boldsymbol{\hat{n}}\,\mathrm{d}S-\rho\int_S\nabla\phi\cdot\delta\boldsymbol{\theta}\frac{\partial\phi}{\delta n}\mathrm{d}S. \qquad \text{(A 3)}$$

The relation (A 3) is equivalent to (3.6) because the displacement $\delta\boldsymbol{\theta}$ of S is normal to S, i.e. we have $\delta\boldsymbol{\theta} = \delta\boldsymbol{\theta}\boldsymbol{\hat{n}}$.

We are grateful to the referee who suggested a simpler proof for (3.5).

REFERENCES

BENJAMIN, T. B. & OLVER, P. J. 1982 Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.* **125**, 137–185.

BRANCHER, J. P. 1980 Sur l'hydrodynamique des ferrofluides. Thèse, University of Nancy.

COWLEY, M. D. & ROSENSWEIG, R. E. 1967 The interfacial stability of a ferromagnetic fluid. *J. Fluid Mech.* **30**, 671–688.

JOFFRE, G. H. 1984 Profils de ménisques électrisés avec ou sans émission de charge. Thèse, University of Paris VI.

LUKE, J. C. 1977 A variational principle for a fluid with a free surface. *J. Fluid Mech.* **27**, 395–397.

Mestel, A. J. 1983 Magnetic levitation of liquid metals. Ph.D. thesis, University of Cambridge.

Milder, D. M. 1977 A note regarding 'On Hamilton's principle for surface waves'. *J. Fluid Mech.* **83**, 159–161.

Miles, J. W. 1977 On Hamilton's principle for surface waves. *J. Fluid Mech.* **83**, 153–158.

Rosensweig, R. E. 1985 *Ferrohydrodynamics*. Cambridge University Press.

Sero-Guillaume, O. 1983 Sur l'équilibre des ferrofluides et des métaux liquides. Thèse, University of Nancy, France.

Sneyd, A. D. & Moffatt, H. K. 1982 Fluid dynamical aspects of the levitation melting process. *J. Fluid Mech.* **117**, 46–70.